6

# RELATIONSHIP BETWEEN WIND VELOCITY AND TEMPERATURE IN THE FREE ATMOSPHERE *

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**ABSTRACT.** In this paper, a number of formulae have been given for the computation of upper air temperature from the known distribution of velocities of upper winds as given by pilot balloon observations. Sir Napier Shaw's formulae giving the relationship between temperature and velocity hold only when geostrophic conditions are satisfied. They are not applicable to tropical regions. For tropical regions, a closer approximation is obtained by the use of the formulæ

$$\frac{g}{T^2}\frac{\partial T}{\partial x} = -2\varpi \cos\phi \frac{\partial}{\partial x}\left(\frac{u}{T}\right) + 2\varpi \sin\phi \frac{\partial}{\partial z}\left(\frac{v}{T}\right),$$

$$\frac{g}{T^2}\frac{\partial T}{\partial y} = -2\varpi \cos\phi \frac{\partial}{\partial y}\left(\frac{u}{T}\right) - 2\varpi \sin\phi \frac{\partial}{\partial z}\left(\frac{u}{T}\right),$$

where T is the temperature in absolute scale, $u$, $v$ the horizonal components of wind towards east and north, $\varpi$ the angular velocity of earth's rotation, and $\phi$ the latitude.

1. Sir Napier Shaw's formulae giving the relationship between temperature and velocity are frequently used to compute upper air temperature from the known distribution of velocities of upper winds as given by pilot balloon observations. These formulae assume geostrophic relationship and do not therefore hold for tropical regions. The best formulae to be used for such computations in tropical regions received the consideration of the present writer some time ago. Starting from the rigorous hydrodynamical equations, a series of formulae were worked out, with the underlying assumptions clearly defined. In this paper, these formulae have been given, and the best formulae for tropical regions have been indicated. The results of certain computations made of the upper air temperature in India from the pilot balloon winds and their comparison with actual observed temperature will be published in a separate paper.

2. Let the axes refer to east, north and vertical respectively, and let $u$, $v$, $w$ be the components of wind velocity at $(x, y, z)$.

The equations of motion are—

$$\frac{Du}{Dt} + 2\varpi(w \cos\phi - v \sin\phi) = -\frac{1}{\rho}\frac{\partial p}{\partial x} + K\frac{\partial^2 u}{\partial z^2}$$

$$\frac{Dv}{Dt} + 2\varpi u \sin\phi = -\frac{1}{\rho}\frac{\partial p}{\partial y} + K\frac{\partial^2 v}{\partial z^2}$$

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$$\frac{Dw}{Dt} - 2\varpi u \cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial q^2}{\partial x} + \eta w - \zeta v, \quad \text{etc.,}$$

where $$q^2 = u^2 + v^2 + w^2,$$

and $$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \; \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \; \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

are the components of vorticity vector. $\phi$ denotes the latitude, $\varpi$ the angular velocity of rotation of the earth, and K the coefficient of eddy viscosity.

3. If we write

$$\beta = 2\varpi \cos\phi, \quad \gamma = 2\omega \sin\phi,$$

the equations of motion become

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial q^2}{\partial x} + (\eta + \beta)w - (\zeta + \gamma)v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + K\frac{\partial^2 u}{\partial z^2} \quad \ldots \quad (1)$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial q^2}{\partial y} + (\zeta + \gamma)u - \xi w = -\frac{1}{\rho}\frac{\partial p}{\partial y} + K\frac{\partial^2 v}{\partial z^2} \quad \ldots \quad (2)$$

$$\frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial q^2}{\partial z} + \xi v - (\eta + \beta)u = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \quad \ldots \quad (3)$$

4. If the motion is steady

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0 \quad \ldots \quad (4)$$

But in the atmosphere, we seldom see a steady state. To take account of the non-steady state, we can assume either

$$u = \lambda t + u_0, \; v = \mu t + v_0, \; w = \nu t + w_0 \quad \ldots \quad (5)$$

or

$$u = u_0 e^{\lambda t}, \; v = v_0 e^{\mu t}, \; w = w_0 e^{\nu t}, \quad \ldots \quad (6)$$

where $\lambda$, $\mu$, $\nu$ are functions of $x$, $y$, $z$.

5. Taking $p = R\rho T$, where T is the temperature, we get, by eliminating $p$ from the equations (1), (2), and (3), the following results on the second assumption, namely (6) :—

$$\frac{g}{T^2}\frac{\partial T}{\partial x} = K\frac{\partial}{\partial z}\left(\frac{1}{T}\frac{\partial^2 u}{\partial z^2}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{T}\frac{\partial q^2}{\partial z}\right) - \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{1}{T}\frac{\partial q^2}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{(\eta + \beta)u - \xi v - \nu w}{T}\right)$$

$$-\frac{\partial}{\partial z}\left(\frac{\lambda u - (\zeta + \gamma)v + (\eta + \beta)w}{T}\right) \quad \ldots \quad (7)$$

$$\frac{g}{T^2}\frac{\partial T}{\partial y}=K\frac{\partial}{\partial z}\left(\frac{1}{T}\frac{\partial^2 v}{\partial z^2}\right)+\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{1}{T}\frac{\partial q^2}{\partial z}\right)-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{1}{T}\frac{\partial q^2}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{(\eta+\beta)u-\xi v-\nu w}{T}\right)$$
$$-\frac{\partial}{\partial z}\left(\frac{(\zeta+\gamma)u+\mu v-\xi w}{T}\right) \quad \ldots \quad (8)$$

If we make the first assumption, namely (5), the equations are obtained by writing $\lambda$, $\mu$, $\nu$ for $\lambda u$, $\mu v$, $\nu w$ respectively in the terms involving the operator

$$\frac{\partial}{\partial t}.$$

6. If the air-mass is supposed to move adiabatically, we have

$$\frac{D}{Dt}\left(\frac{T^{\alpha}}{p^{\alpha-1}}\right)=0,$$

$\alpha$ being the ratio of specific heats,

or,
$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\frac{\alpha-1}{\alpha}\frac{T}{p}\left(\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right) \quad \ldots \quad (9)$$

If the motion is along an isotherm,

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=0 \quad (10)$$

If the motion is along an isentropic surface that is, the surfaces $\Phi=$constant, we get

$$u\frac{\partial \phi}{\partial x}+v\frac{\partial \phi}{\partial y}+w\frac{\partial \phi}{\partial z}=0,$$

or,
$$\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)-\frac{T}{p}\frac{\alpha-1}{\alpha}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}\right)=0. \quad \ldots \quad (11)$$

The two equations (7) and (8), coupled with any of the above, as the circumstances may be, may be used to calculate

$$\frac{\partial T}{\partial x},\ \frac{\partial T}{\partial y},\ \frac{\partial T}{\partial z},$$

if we know the velocity distribution.

These are very general equations, and are not easily amenable to numerical computation.

7. If the fluid elements move along stream lines in such a way that the angular rotation can be neglected, we get

$$\xi=0,\ \eta=0,\ \zeta=0.$$

If in addition, the viscosity is neglected, we get

$$\frac{g}{T^2}\frac{\partial T}{\partial x}=\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{T}\frac{\partial q^2}{\partial x}\right)-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{1}{T}\frac{\partial q^2}{\partial x}\right)-\frac{\partial}{\partial x}\frac{\beta u-\nu w}{T}-\frac{\partial}{\partial z}\frac{\lambda u-\gamma v+\beta w}{T} \quad (12)$$

$$\frac{g}{T^2}\frac{\partial T}{\partial y}=\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{1}{T}\frac{\partial q^2}{\partial z}\right)-\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{1}{T}\frac{\partial q^2}{\partial y}\right)-\frac{\partial}{\partial y}\frac{\beta u-\nu w}{T}-\frac{\partial}{\partial z}\frac{\gamma u+\mu v}{T}. \quad \ldots \quad (13)$$

8. *Sir Napier Shaw's equations.*

If the motion is steady,

$$\lambda=\mu=\nu=0,$$

and if the vertical velocity and the square of the velocity can be neglected,

$$w=0,\quad q=0.$$

On these assumptions, we get

$$\frac{g}{T^2}\frac{\partial T}{\partial x}=-\beta\frac{\partial}{\partial x}\left(\frac{u}{T}\right)+\gamma\frac{\partial}{\partial z}\left(\frac{v}{T}\right). \quad \ldots \quad (14)$$

$$\frac{g}{T^2}\frac{\partial T}{\partial y}=-\beta\frac{\partial}{\partial y}\left(\frac{u}{T}\right)-\gamma\frac{\partial}{\partial z}\left(\frac{u}{T}\right). \quad \ldots \quad (15)$$

These two equations reduce to Sir Napier Shaw's equations* when $\beta=0$, that is, $2\bar{\omega}\cos\phi=0$.

All these collectively constitute very large assumptions which are not justified, particularly in tropical regions. It is, therefore, little wonder that the correlation coefficient between the observed variations of temperature and those deduced by use of Sir Napier Shaw's equations has been found to be low, little less than 0.5, even in high latitudes. In latitudes near the equator, the assumption $\cos\phi=0$ completely invalidates the equations. Moreover, the vertical velocity, even though small compared with the horizontal, makes a very large contribution to temperature variation, and cannot, therefore, be neglected in such equations.

9. At latitude 25°,

$$\beta=2\times7\times10^{-5}\times0.9 \text{ cm./s}$$
$$\gamma=2\times7\times10^{-5}\times0.4 \text{ cm./s.}$$

In latitudes near the equator, Shaw's equations are approximately true, provided $u/T$ undergoes no variation in either horizontal direction, that is, $u/T$ remains constant in the entire horizontal field.

If the motion is north-south, then

$$\frac{g}{T^2}\frac{\partial T}{\partial x}=\gamma\frac{\partial}{\partial z}\left(\frac{v}{T}\right). \quad \ldots \quad (16)$$

The temperature gradient is in the east-west direction; temperature is increasing or decreasing eastwards according as the ratio of velocity to temperature is increasing or decreasing upwards.

If the motion is east-west, we get

$$\frac{g}{T^2}\frac{\partial T}{\partial x}=-\beta\frac{\partial}{\partial x}\left(\frac{u}{T}\right)$$

or

$$\frac{g}{T}=\frac{\beta u}{T}+\text{const.},$$

that is,

$$u=AT+B. \quad \ldots \quad (17)$$

* Manual of Meteorology, *see* reference 2 at end.

In a purely east-west motion, therefore, the temperature and velocity become linearly related along the same east-west line. That is, if velocity is constant along a line, temperature is also constant along that line. In illustration of this, see the upper wind and temperature charts for all levels, 2 km. and above, for the months November to March, in the Memoir on the General Circulation of the atmosphere over India and its neighbourhood by Dr. Ramanathan and Mr. Ramakrishnan (1939).

10. If we assume the motion to be steady, that is $\lambda=\mu=\nu=0$, we get, on simplifying the generalised equations (7) and (8),

$$\left(g-\beta u+\tfrac{1}{2}\frac{\partial q^2}{\partial z}\right)\frac{\partial T}{\partial x}+\left(\gamma v-\beta w-\tfrac{1}{2}\frac{\partial q^2}{\partial x}\right)\frac{\partial T}{\partial z}=T\left(\beta\frac{\partial v}{\partial y}+\gamma\frac{\partial v}{\partial z}\right) \quad (18)$$

and

$$\left(g-\beta u+\tfrac{1}{2}\frac{\partial q^2}{\partial z}\right)\frac{\partial T}{\partial y}+\left(\gamma u-\tfrac{1}{2}\frac{\partial q^2}{\partial y}\right)\frac{\partial T}{\partial z}=T\left(-\beta\frac{\partial u}{\partial y}+\gamma\frac{\partial u}{\partial z}\right) \quad \ldots \quad (19)$$

If $w=0$, and if the velocities do not undergo variation in the horizontal direction,

$$\left(g-\beta u+\tfrac{1}{2}\frac{\partial q^2}{\partial z}\right)\frac{\partial T}{\partial x}+\gamma v\frac{\partial T}{\partial z}=T\gamma\frac{\partial v}{\partial z} \quad \ldots \quad (20)$$

$$\left(g-\beta u+\tfrac{1}{2}\frac{\partial q^2}{\partial z}\right)\frac{\partial T}{\partial y}+\gamma u\frac{\partial T}{\partial z}=T\gamma\frac{\partial u}{\partial z} \quad \ldots \quad (21)$$

or,

$$\left(g-\beta u+\tfrac{1}{2}\frac{\partial q^2}{\partial z}\right)\left(\frac{\partial T}{\partial x}+\frac{\partial T}{\partial y}\right)+\gamma(u+v)\frac{\partial T}{\partial z}=T\gamma\frac{\partial}{\partial z}(u+v) \quad \ldots \quad (22)$$

or,

$$\left(g-\beta u+\tfrac{1}{2}\frac{\partial q^2}{\partial z}\right)\times\text{(mean horizontal lapse rate)}$$

$$+\gamma\frac{u+v}{2}\times\text{(vertical lapse rate)}$$

$$=T\gamma\frac{\partial}{\partial z}\frac{u+v}{2}. \quad \ldots \quad (23)$$

If velocity does not vary with height,

$$\frac{\text{mean hor. lapse rate}}{\text{vertical lapse rate}}=-\frac{(u+v)\gamma}{2(g-\beta u)} \quad (24)$$

If $u=v=20m/s=72$ km/hr, we have at $\phi=25°$,

$$\frac{\text{mean hor. lapse rate}}{\text{vertical lapse rate}}=-\frac{0.112}{981-0.252},$$

When there is no horizontal or vertical variation of wind velocity, the horizontal lapse rate is a very small fraction of the vertical lapse rate. For adiabatic lapse rate in the vertical, the horizontal lapse rate becomes 1.2°C per 1000 km., under the conditions stipulated.

11. In a region where the gradient equation holds,

$$\frac{1}{\rho}\frac{\partial p}{\partial n}=2\bar{\omega}\,V\sin\phi\pm\frac{V^2}{r}, \qquad \ldots \quad (25)$$

$\partial n$ being an element of normal to the isobar.

Also,

$$\frac{\partial p}{\partial z}=-g\rho,\; p=R\rho T,\; \therefore \frac{1}{p}\frac{\partial p}{\partial z}=-\frac{g}{RT}, \qquad \frac{g}{RT^2}\frac{\partial T}{\partial n}=\frac{\partial}{\partial n}\left(\frac{1}{p}\frac{\partial p}{\partial z}\right)$$

$$=\frac{\partial^2}{\partial n\partial z}\log p=\frac{\partial^2}{\partial z\partial n}\log p=\frac{\partial}{\partial z}\left(\frac{1}{p}\frac{\partial p}{\partial n}\right)=\frac{\partial}{\partial z}\left(\frac{2\omega V\sin\phi}{RT}\pm\frac{V^2}{RTr}\right).$$

Therefore,

$$\frac{1}{T^2}\frac{\partial T}{\partial n}=\frac{2\bar{\omega}\sin\phi}{g}\cdot\frac{\partial}{\partial z}\left(\frac{V}{T}\right)\pm\frac{1}{g}\frac{\partial}{\partial z}\left(\frac{V^2}{rT}\right). \qquad \ldots \quad (26)$$

$$\bar{\omega}=7\times10^{-5}\text{ cm./sec., and at }\phi=25^\circ,\ \sin\phi=0.4$$

Therefore,

$$\frac{1}{T^2}\frac{\partial T}{\partial n}=\frac{5.6\times10^{-10}}{981}\left(\frac{V_2}{T_2}-\frac{V_1}{T_1}\right)\pm\frac{10^{-5}}{981}\left(\frac{V_2^2}{r_2T_2}-\frac{V_1^2}{r_1T_1}\right),$$

where $V_1$, $T_1$, $r_1$ and $V_2$, $T_2$, $r_2$ are velocity, temperature, and radius of curvature of isobar at height $z$ and one kilometre higher up, $\partial n$ being measured in units of kilometre.

If $$r_1=r_2=100\text{ km.}=10^7\text{ cm},$$

$$V_1=15m./s.=53\text{ km./hr.}, \qquad T_1=276^\circ\text{A}.$$

$$V_2=20m./s.=72\text{ km./hr.}, \qquad T_2=270^\circ\text{A}.,$$

we get

$$\frac{\partial T}{10^{-5}\partial n}=\begin{array}{c}0.05\text{ for cyclone,}\\ \text{and}\\ -0.03\text{ for anticyclone.}\end{array}$$

With these data temperature is increasing outwards at the rate of 5°C per 100 km., in the case of cyclone, and decreasing outwards at the rate of 3°C in the case of anticyclone.

If, in a cyclone, velocity decreases upwards in such a way that $\frac{V_2}{T_2}<\frac{V_1}{T_1}$,

and, therefore, also $\frac{V_2^2}{T_2}<\frac{V_1^2}{T_1}$, we have $\frac{\partial T}{\partial n}$ negative, and therefore, temperature increasing towards the centre. Or, air on the axis is at a higher temperature than the surrounding air.

Reverse condition holds when the velocity increases upwards so that

$$\frac{V_2}{T_2} > \frac{V_1}{T_1}, \text{ and, therefore, also } \frac{V_2^2}{T_2} > \frac{V_1^2}{T_1}.$$

In an anticyclone, if velocity increases or decreases fairly rapidly with height, then the term involving

$$\frac{V_2^2}{T_2} - \frac{V_1^2}{T_1}$$

is the dominant term. In this case the axis is cold or warm according as the velocity is decreasing or increasing upwards.

12. For the computation of temperature from the distribution of winds in tropical regions, a better approximation is obtained by the use of the equations

$$\frac{g}{T^2}\frac{\partial T}{\partial x} = -\beta\frac{\partial}{\partial x}\left(\frac{u}{T}\right) + \gamma\frac{\partial}{\partial z}\left(\frac{v}{T}\right)$$

$$\frac{g}{T^2}\frac{\partial T}{\partial y} = -\beta\frac{\partial}{\partial y}\left(\frac{u}{T}\right) - \gamma\frac{\partial}{\partial z}\left(\frac{u}{T}\right).$$

than by Sir Napier Shaw's equations, because the assumption that $2\omega \cos \phi = 0$, which does not hold in the tropical region, is not made.

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REFERENCES

[1] Ramanathan and Ramakrishnan, 1939, *Ind. Met. Memoirs*, **26**, Part 10.

[2] Shaw, Sir N., 1931, Manual of eteorology, Vol. IV, Ch. 7.